Pseudorandom sequences derived from automatic sequences

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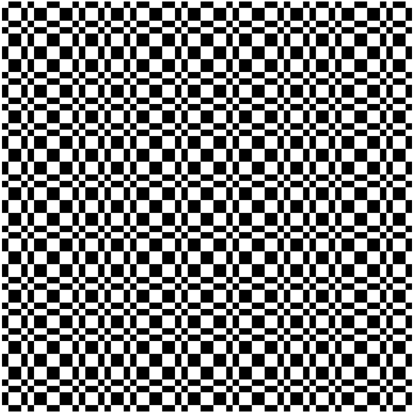
Is this random?



... or is this more random?

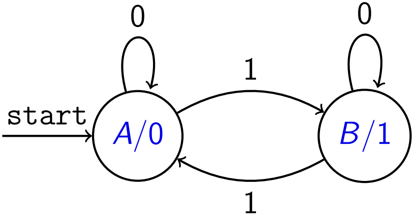
Pseudorandom sequences

- generated by deterministic algorithms which simulate randomness
- not random at all but guarantee certain desirable features (depending on application)
- mathematical/cryptographic point of view: as many desirable features as possible
- many different measures: linear complexity, correlation, normality, ...

Thue-Morse sequence

$$t_0 = 0, \quad t_n = \begin{cases} t_{n/2} & \text{if } n \text{ is even,} \\ 1 - t_{(n-1)/2} & \text{if } n \text{ is odd,} \end{cases} \qquad n = 1, 2, \ldots$$

- t_n is the sum of digits of n modulo 2, $n = 0, 1, \ldots$
- automatic sequence generated by the Thue-Morse automaton



- $t_0 \ldots t_{11} = 011010011001\ldots$
  $t_{11} = 1 - t_5 = t_2 = t_1 = 1 - t_0 = 1$
  or $11 = 8 + 2 + 1 = (1011)_2$: $t_{11} = 1 + 0 + 1 + 1 \equiv 1 \bmod 2$

The first 4096 sequence elements

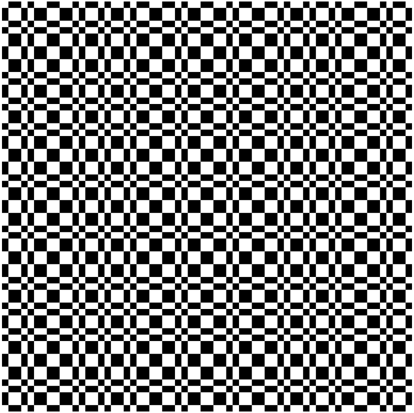


Figure: The first 4096 elements of the Thue-Morse sequence split into 64 rows of each 64 sequence elements. Zeros are represented by white, ones are represented by black.

Features

pseudorandom/desirable:

- large Nth linear complexity
- large Nth maximum-order complexity
- balance
- small well-distribution measure

not pseudorandom/undesirable:

- very large correlation measure of order 2
- very small expansion complexity
- there are short patterns such as 000 and 111 which do not appear in the sequence
- subword complexity is only linear

Subsequences

- may destroy the non-random structure of the original sequence
- may keep the desirable features of pseudorandomness

promising candidates:

- along squares, cubes, bi-squares, ... or along the values of any polynomial f of degree at least 2 with $f(\mathbb{N}_0) \subset \mathbb{N}_0$
- along primes
- along the Piatetski-Shapiro sequence $\lfloor n^c \rfloor$, $1 < c < 2$,
- along geometric sequences such as 3^n

Thue-Morse sequence along squares

inherits:

- large maximum-order complexity and thus a large linear complexity
- asymptotically balanced/simply normal

in contrast to the original sequence:

- unbounded expansion complexity
- normal, that is, asymptotically each pattern appears with the right frequency in the sequence

Roughly speaking: looks much more random than the original sequence.

Figure: The first 4096 elements of the Thue-Morse sequence along squares split into 64 rows of each 64 sequence elements. Zeros are represented by white, ones are represented by black.

Linear complexity

The Nth linear complexity $L((s_n), N)$ of a sequence (s_n) over $\mathbb{F}_2 = \{0, 1\}$ is the length L of a shortest linear recurrence relation satisfied by the first N elements of (s_n),

$$s_{n+L} \equiv c_{L-1} s_{n+L-1} + \cdots + c_1 s_{n+1} + c_0 s_n \bmod 2, \quad 0 \leq n \leq N - L - 1,$$

for some $c_0, \ldots, c_{L-1} \in \mathbb{F}_2$.
linear complexity: $L((s_n)) = \sup_{N \geq 1} L((s_n), N)$

- expected value $\frac{N}{2} + O(1)$ (Gustavson, 1976)
- deviations of magnitude $\log N$ must appear for infinitely many N (Niederreiter, 1988)
- $L((s_n)) < \infty \iff (s_n)$ ultimately periodic

Linear complexity of Thue-Morse sequence

$$L((t_n), N) = 2 \left\lfloor \frac{N+2}{4} \right\rfloor \quad \text{(Mérai/W., 2018)}$$

Proof of $L((t_n), N) \geq \frac{N-1}{2}$:

- (t_n) is not (ultimately) periodic
- generating function $G(x) = \sum_{n=0}^{\infty} t_n x^n$ is not rational
- $G(x)$ is algebraic over $\mathbb{F}_2(x)$:
  $h(x, G(x)) := (x+1)^3 G(x)^2 + (x+1)^2 G(x) + x = 0$

$$\sum_{\ell=0}^{L} c_\ell t_{n+\ell} = 0 \quad \text{for } 0 \le n \le N-L-1$$

$$f(x) = \sum_{\ell=0}^{L} c_\ell x^{L-\ell} \quad \text{and} \quad g(x) = \sum_{m=0}^{L-1} \left(\sum_{\ell=L-m}^{L} c_\ell t_{m+\ell-L} \right) x^m$$

$$f(x)G(x) \equiv g(x) \bmod x^N$$

$$\begin{aligned} f(x)^2 h(x, g(x)/f(x)) &= (x+1)^3 g(x)^2 + (x+1)^2 f(x) g(x) + x f(x)^2 \\ &= K(x) x^N, \quad K(x) \neq 0 \end{aligned}$$

$$2L+1 \ge N$$

Christol's theorem

Let

$$G(x) = \sum_{n=0}^{\infty} s_n x^n$$

be the generating function of the sequence (s_n) over $\mathbb{F}_q$. Then (s_n) is q-automatic if and only if $G(x)$ is algebraic over $\mathbb{F}_q(x)$, that is, there is a polynomial $h(x,y) \in \mathbb{F}_q[x,y] \setminus \{0\}$ such that $h(x, G(x)) = 0$.

Linear complexity of automatic sequences

Mérai/W., 2018:

Let q be a prime power and (s_n) be a q-automatic sequence over $\mathbb{F}_q$ which is not ultimately periodic. Let $h(x,y) = h_0(x) + h_1(x)y + \cdots + h_d(x)y^d \in \mathbb{F}_q[x,y]$ be a non-zero polynomial $h(x, G(x)) = 0$ with no rational function $r(x) \in \mathbb{F}_q(x)$ satisfying $h(x, r(x)) = 0$.

Put

$$M = \max_{0 \le i \le d} \{\deg h_i - i\}.$$

Then we have

$$\frac{N-M}{d} \le L((s_n), N) \le \frac{(d-1)N + M + 1}{d}.$$

- Upper bound comes from (Berlekamp-Massey algorithm) $L((s_n), N+1) \in \{L((s_n), N), N+1-L((s_n), N)\}$.

- (not ultimately periodic) automatic sequences have large Nth linear complexity
- Thue-Morse sequence: $L((t_n), N) = \frac{N}{2} + O(1)$
- never $\left|L((t_n), N) - \frac{N}{2}\right| \approx \log N$
- idea does not work for (t_{n^2})

Next we study a finer measure than linear complexity.
As a consequence we get

$$L((t_{n^2}), N) \geq cN^{1/2}$$

for some $c > 0$.

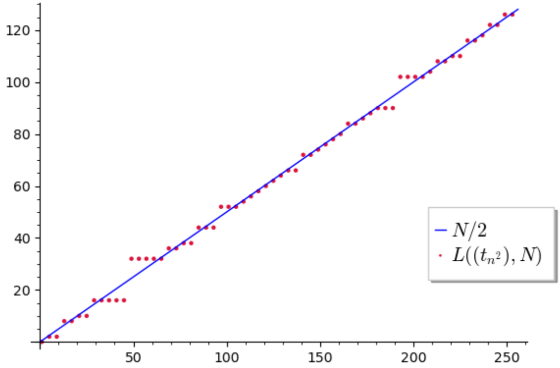


Figure: Nth linear complexity of the Thue-Morse sequence along squares.

Problem

Prove that

$$L((t_{n^2}), N) = \frac{N}{2} + o(N).$$

Maximum order complexity

The Nth maximum order complexity $M((s_n),N)$ is the smallest positive integer M with

$$s_{n+M} = f(s_{n+M-1},\ldots,s_n), \quad 0 \le n \le N-M-1,$$

for some mapping $f : \mathbb{F}_2^M \to \mathbb{F}_2$.

- $L((s_n),N) \ge M((s_n),N)$
- Jansen, 1990: expected value $\approx \log N$
- $(s_n, s_{n+1},\ldots,s_{n+M-2}) = (s_m, s_{m+1},\ldots,s_{m+M-2})$, $s_{n+M-1} \ne s_{m+M-1}$ for some $0 \le n < m \le N-M$ $\Longrightarrow M((s_n),N) \ge M$
- $C_2((s_n),N) \ge M((s_n),N) - 1$
- Chen/Gomez/Gomez/Tirkel, 2022: $C_2((s_n),N) \ge N - 2^{M((s_n),N)} + 1$
- desirable: $\log N \ll M((s_n),N) = o(N)$

Maximum order complexity of Thue-Morse sequence

Sun/W., 2019:

$$M((t_n), N) = 2^\ell + 1, \quad \text{where} \quad \ell = \left\lceil \frac{\log(N/5)}{\log 2} \right\rceil, \quad N \geq 4.$$

- $\frac{N}{5} + 1 \leq M((t_n), N) \leq 2\frac{N-1}{5} + 1$
- proof with Walnut (J. Shallit, personal communication)
- $M((t_n), N)$ too large: implies very large correlation measure of order 2 (aperiodic autocorrelation)

Maximum order complexity of Thue-Morse sequence along squares

Sun/W., 2019:

$$L((t_{n^2}),N) \ge M((t_{n^2}),N) \ge \sqrt{\frac{2N}{5}}, \quad N \ge 21.$$

- similar bounds for Rudin-Shapiro sequence, pattern sequences with the all 1 pattern
  (Rudin-Shapiro sequence (r_n):

$$r_n = \sum_{i=0}^{\infty} n_i n_{i+1}, \quad n = \sum_{i=0}^{\infty} n_i 2^i, \; n_i \in \{0,1\})$$

- Popoli, 2020: extension to polynomials of degree ≥ 2
  lower bound of order of magnitude $N^{1/d}$

Correlation measure of order k

For $k \geq 1$, the Nth correlation measure of order k of a binary sequence (s_n) is

$$C_k((s_n), N) = \max_{M,D} \left| \sum_{n=0}^{M-1} (-1)^{s_{n+d_1}} \cdots (-1)^{s_{n+d_k}} \right|,$$

where the maximum is taken over all $D = (d_1, d_2, \ldots, d_k)$ with integers satisfying $0 \leq d_1 < d_2 < \cdots < d_k$ and $1 \leq M \leq N - d_k$.

- introduced by Mauduit and Sárközy, 1997
- Alon et al., 2007: expected value $\Theta\left(\sqrt{N \log \binom{N}{k}}\right)$, $2 \leq k \leq N/4$
- $C_2((s_n), N) \geq M((s_n), N) - 1$
- Mauduit/Sárközy, 1998: $C_2((t_n), N) > \frac{N}{12}$, $N \geq 5$
- $C_2((t_n), N) \geq M((t_n), N) - 1 \geq \frac{N}{5}$

Correlation measure of order 2 of Thue-Morse sequence along squares

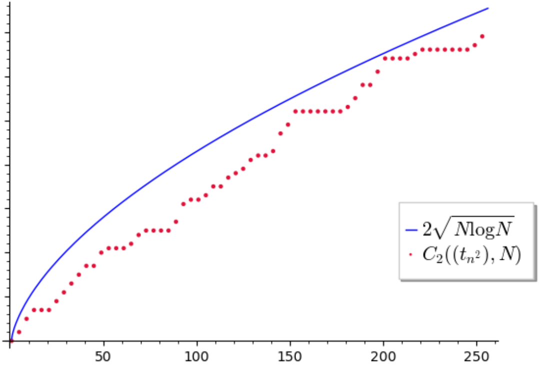


Figure: The Nth second order correlation measure of the Thue-Morse along squares.

Problem

For fixed $k = 2, 3, \ldots$ *show that*

$$C_k((t_{n^2}), N) = o(N).$$

Expansion complexity

Let (s_n) be a sequence over $\mathbb{F}_q$ with generating function

$$G(x) = \sum_{n=0}^{\infty} s_n x^n.$$

For a positive integer N, the Nth expansion complexity $E((s_n), N)$ of (s_n) is $E((s_n), N) = 0$ if $s_0 = \cdots = s_{N-1} = 0$ and otherwise the least total degree of a non-zero polynomial $h(x, y) \in \mathbb{F}_q[x, y]$ such that

$$h(x, G(x)) \equiv 0 \bmod x^N. \qquad (1)$$

$$E((s_n)) = \sup_{N \geq 1} E((s_n), N)$$

is the expansion complexity of (s_n).

Properties

- introduced by C. Diem, 2012
- $E((s_n)) < \infty \iff (s_n)$ is automatic (Christol)
- $E((t_n)) = 5$: $h(x, y) = (x+1)^3 y^2 + (x+1)^2 y + x$
- $E((t_{n^2})) = \infty$
- typical value $E((s_n), N) \approx N^{1/2}$ (Gomez/Mérai, 2020)
- $E((s_n), N) \leq \min\{L((s_n), N) + 1, N - L((s_n), N) + 2\}$ (Mérai/Niederreiter/W., 2017)

Expansion complexity of (t_{n^2})

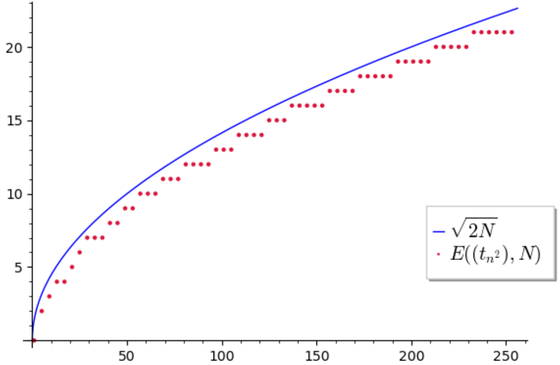


Figure: The Nth expansion complexity of the Thue-Morse sequence along squares.

Subword complexity

For a sequence (s_n) over the alphabet Δ the subword complexity $p((s_n), k)$ is the number of distinct subsequences of length k.

- $1 \le p((s_n), k) \le |\Delta|^k$
- (s_n) automatic (not ultimately periodic): $p((s_n), k)$ is of order of magnitude k

Normality

A sequence (s_n) is called normal if for any fixed length k and any pattern $\mathbf{e} \in \Delta^k$

$$\frac{\#\{0 \le n < N : (s_n, s_{n+1}, \ldots, s_{n+k-1}) = \mathbf{e}\}}{N} \to \frac{1}{|\Delta|^k},$$

as $N \to \infty$.

- (t_{n^2}) is normal (Drmota/Mauduit/Rivat, 2019)
- open: Is $(t_{f(n)})$ normal for $\deg(f) \ge 3$?
- $p((t_{n^2}), k) = 2^k$
- implies small correlation of fixed order with bounded lags $d_1 < \ldots < d_k \le B$

Analogs for finite fields

For a prime p and $q = p^r$ with $r \geq 2$ let $(\beta_1, \ldots, \beta_r)$ be an ordered basis of $\mathbb{F}_q$ over $\mathbb{F}_p$.

Thue-Morse function:

$$T\left(\sum_{i=1}^{r} x_i\beta_i\right) = \sum_{i=1}^{r} x_i, \quad x_1, \ldots, x_r \in \mathbb{F}_p$$

Rudin-Shapiro function

$$R\left(\sum_{i=1}^{r} x_i\beta_i\right) = \sum_{i=1}^{r-1} x_i x_{i+1}, \quad x_1, \ldots, x_r \in \mathbb{F}_p$$

Balance of Thue-Morse function

Dartyge/Sárközy, 2013 (using the Weil bound):
Let $f \in \mathbb{F}_q[x]$ be of degree d with $\gcd(d,q)=1$. Then for all $c \in \mathbb{F}_p$, we have

$$N_c := \left|\#\{\xi \in \mathbb{F}_q : T(f(\xi)) = c\} - p^{r-1}\right| \leq (d-1)p^{r/2}.$$

Sketch of Proof. $T(f(\xi)) = Tr(\underbrace{\sum_{i=1}^{r} \delta_i}_{=:\delta\neq 0} f(\xi))$, $\{\delta_1,\ldots,\delta_r\}$ dual basis

$$N_c = \frac{1}{p}\sum_{a\in\mathbb{F}_p}\sum_{\xi\in\mathbb{F}_q}\underbrace{\psi_p(Tr(a\delta f(\xi))-c)}_{\psi_q(a\delta f(\xi)-\eta)},$$

ψ_u additive canonical character of $\mathbb{F}_u$

Normality

Makhul/W., 2022:
Assume $1 \leq d < p$ and $s \leq d$. For any polynomial $f \in \mathbb{F}_q[x]$ of degree d and any pairwise distinct $\alpha_1, \ldots, \alpha_s \in \mathbb{F}_q$ and any $c_1, \ldots, c_s \in \mathbb{F}_p$ we have

$$\left|\#\{\xi \in \mathbb{F}_q : T(f(\xi + \alpha_i)) = c_i, 1 \leq i \leq s\} - p^{r-s}\right| \leq (d-1)p^{r/2}.$$

Balance of Rudin-Shapiro function

Dartyge/Mérai/W., 2021 (using Hooley-Katz bound):
Let $f \in \mathbb{F}_q[x]$ be of degree d with $\gcd(d, q) = 1$. Then for all $c \in \mathbb{F}_p$, we have

$$\left|\#\{\xi \in \mathbb{F}_q : R(f(\xi)) = c\} - p^{r-1}\right| \leq C_{d,r} p^{(3r+1)/4},$$

where the constant $C_{d,r}$ depends only on the degree d of f and r.

- Weil fails (degree (as univariate polynomial) too large)
- Deligne fails ((multivariate) polynomial has singular points)
- Hooley-Katz is generalization of Deligne for non-singular polynomials

The Hooley-Katz Theorem, 1991

We denote by $\overline{\mathbb{F}_p}$ the algebraic closure of $\mathbb{F}_p$.
The (affine) singular locus $\mathcal{L}(F)$ of a polynomial F over $\mathbb{F}_p$ in r variables is the set of common zeros in $\overline{\mathbb{F}_p}^r$ of the polynomials

$$F, \frac{\partial F}{\partial X_1}, \ldots, \frac{\partial F}{\partial X_r}.$$

Let Q be a polynomial over $\mathbb{F}_p$ in r variables of degree $D \geq 1$ such that the dimensions of the singular loci of Q and its homogeneous part Q_D of degree D satisfy

$$\max\{\dim(\mathcal{L}(Q)), \dim(\mathcal{L}(Q_D)) - 1\} \leq s.$$

Then the number N of zeros of Q in $\mathbb{F}_p^r$ satisfies

$$\left|N - p^{r-1}\right| \leq C_{D,r} p^{(r+s)/2}.$$

$s = -1$: Deligne

Problem

Study the normality of the Rudin-Shapiro function at $f(x)$. *Namely, show that*

$$\frac{\#\{\xi \in \mathbb{F}_q : R(f(\xi+\alpha_i)) = c_i, 1 \le i \le s\}}{p^{r-s}} \to 1 \quad \textit{as } p \to \infty$$

for some $s \geq 2$ *and any* $f \in \mathbb{F}_q[x]$ *of fixed degree.*

Further reading

L. Mérai, A. Winterhof, Pseudorandom sequences derived from automatic sequences. Cryptogr. Commun. 14 (2022), no. 4, 783–815.

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Thank you for your attention!